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THE EFFECT OF HIGH-FREQUENCY VIBRATION ON THE ONSET OF MARANGONI CONVECTION IN A HORIZONTAL LIQUID LAYER†

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The effect of high-frequency translational harmonic vibrations on the onset of thermocapillary convection in a horizontal liquid layer, bounded above by a free surface and below by a solid wall, is investigated by averaging the convection equations. It is shown that longitudinal vibrations have no effect on convective instability. If the direction of the vibration contains a transverse component, and the action of the vibration has a stabilizing effect: the free boundary of a uniform liquid is smoothed and thermal convection of a non-uniform liquid may be suppressed. The maximum stabilizing effect is obtained for vertical vibrations.

The problem of the effect of high-frequency vertical vibrations on the onset of convection in a region with a solid boundary was considered previously in [1], and the method of averaging was used to derive a closed autonomous system for the averaged hydrodynamic field. It has been shown [1, 2], that vertical vibrations hinder the onset of thermogravitational convection in a horizontal liquid layer and can even give rise to a state of relative equilibrium that is absolutely stable. In the case of vibrations in an arbitrary direction, it has been shown [3],‡ that vibrations can have both a stabilizing and a destabilizing effect; it has been established, for example, that for all directions of vibration, differing from the vertical direction, gravitational convection can occur in a liquid layer not only when it is heated from below but also from above. The averaged equations [3] were analysed in [4, 5] for the interesting special case of weightlessness conditions. An experimental confirmation of vibration effects [1–5] were given in [6, 7]. The basis of the averaging method for convection problems in the region of a solid boundary was presented in [8, 9]. The method of averaging has been developed [10] for dynamical systems with constraints.

A number of publications have been devoted to investigating thermocapillary convection (see the review [11]).

The problem of Oberbek–Boussinesq thermocapillary convection in a liquid layer with a free non-deformable boundary in the case of vertical vibrations was considered for the first time by Briskman [12], and was investigated further in [13]. The following approach was used to investigate vibration convection in regions with a free boundary [14, 15]: the initial equations were written in general form, averaging was carried out, and a changeover was then made to the Oberbek–Boussinesq equations. This approach was then used [16]§ to investigate convection in a horizontal layer with a deformable free boundary; it was shown that in the case of a slightly non-isothermal liquid the general Oberbek–Boussinesq equations can be taken as the initial equations, i.e. the density is assumed to be variable not only in the mass forces but also in the inertial terms.

In this paper we investigate the effect of translational harmonic vibrations in an arbitrary direction on the thermocapillary instability in a thin horizontal layer of a viscous incompressible liquid, bounded by a deformable free boundary and a solid wall. The Krylov–Bogolyubov averaging method is applied to the generalized Oberbek–Boussinesq convection equations, on the assumption that the vibration frequency ω is high and the velocity amplitude a is finite. An equilibrium solution of the average

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§See also: Zen'kovskaya, S. M. and Shleikel', A. L., Marangoni convection in a high-frequency vibration field. Rostov University, Rostov-on-Don, 2000. Deposited at the All-Union Institute for Scientific and Technical Information, 6 June 2000, No. 1615–V00.

problem is obtained and its stability is investigated. The cases when the liquid is uniform and the free boundary is deformable are considered, and also the case when the liquid is non-uniform and the free boundary is, on average, non-deformable. The interaction between the thermocapillary and the thermo-gravitational mechanisms of convective instability is investigated.

1. FORMULATION OF THE PROBLEM

Consider an infinite horizontal layer of viscous incompressible heat-conducting liquid, bounded below by a solid wall $x_3 = h$ and bounded above by a free deformable surface $x_3 = \xi'(x_1, x_2, t)$. Heat-exchange conditions of general form are specified on each of the boundaries. We will assume that the mean thickness of the layer h is fairly small, so that the equation of state can be taken in the form

$$\rho = \rho_0(1 - \beta(T' - T_0'))$$

Surface tension forces with coefficient $\sigma = \sigma_0 - \sigma_T(T' - T_0')$ act on the free boundary, where $\sigma_T = |\partial\sigma/\partial T'|$. Above the liquid layer there is a gas, whose density is negligibly small and whose temperature and pressure are constant. It is assumed that the layer as a whole undergoes plane translational harmonic vibrations in the direction of the vector $\mathbf{s} = (\cos\varphi, 0, \sin\varphi)$ which vary as $(a/\tilde{\omega})\cos\tilde{\omega}t$, where φ is the angle of the direction of the vibration, so that $\varphi = 0$ corresponds to horizontal vibrations while $\varphi = \pi/2$ corresponds to vertical vibrations. We will choose the system of coordinates in such a way that the x_3 axis coincides with the direction of the gravity force, while the origin of coordinates is taken on the unperturbed free boundary. The x_3 coordinate will henceforth sometimes be denoted by z.

Suppose $\mathbf{v}'$ is the relative velocity vector, T' is the temperature, measured from a certain constant value T_0', p' is the deviation from atmospheric pressure, ρ_0 is the density, β is the coefficient of thermal expansion, ν, χ are the coefficient of kinematic viscosity and the thermal diffusivity, $\mathbf{g} = \gamma g$, where $\boldsymbol{\gamma} = (0, 0, 1)$ is the unit vector of the x_3 axis, g is the acceleration due to gravity, $\mathbf{w}_e = -a\tilde{\omega}\cos\tilde{\omega}t \cdot \mathbf{s}$ is the transfer acceleration, and a is the amplitude of the vibration velocity. We will change to dimensionless variables by choosing as the scales of length, time, velocity, pressure and temperature gradient h, h^2/ν, ν/h, $\rho_0\nu^2/h^2$, A respectively, and we will denote the dimensionless variables by the same letters as the dimensional variables.

In a moving system of coordinates connected with the vibrating layer, we will write the convection equations in the form

$$(1 - \varepsilon T')\frac{d\mathbf{v}'}{dt} = -\nabla p' + \Delta\mathbf{v}' + (1 - \varepsilon T')(\mathrm{Ga}\,\boldsymbol{\gamma} + \omega\,\mathrm{Re}\cos\omega t\,\mathbf{s}) \tag{1.1}$$

$$\operatorname{div}\mathbf{v}' = 0, \quad \frac{dT'}{dt} = \mathrm{Pr}^{-1}\,\Delta T'$$

System (1.1) differs from the Oberbek–Boussinesq convection equations by the term $\varepsilon T' d\mathbf{v}'/dt$. The use of a more general initial model for a slightly non-isothermal liquid ($\varepsilon \to 0$) does not lead to any change in the results.

The following conditions must be satisfied on the free boundary $x_3 = \xi'(x_1, x_2, t)$

$$(\mathbf{v}', \mathbf{l}') = \frac{\partial\xi'}{\partial t}, \quad \mathbf{l}' = \left(-\frac{\partial\xi'}{\partial x_1}, -\frac{\partial\xi'}{\partial x_2}, 1\right) \tag{1.2}$$

$$\tau_{ik}' n_k' - p' n_i' = 2\left(C - \frac{\mathrm{Ma}}{\mathrm{Pr}}T'\right)Kn_i' - \frac{\mathrm{Ma}}{\mathrm{Pr}}\frac{\partial T'}{\partial x_i}, \quad i = 1, 2 \tag{1.3}$$

$$\tau_{3k}' n_k' - p' n_3' = 2\left(C - \frac{\mathrm{Ma}}{\mathrm{Pr}}T'\right)Kn_3', \quad \tau_{ik}' = \frac{\partial v_i'}{\partial x_k} + \frac{\partial v_k'}{\partial x_i}, \quad k = 1, 2, 3$$

$$\frac{\partial T'}{\partial n'} - \mathrm{Bi}\,T' = \delta_1 \tag{1.4}$$

Here τ_{ik}' are the components of the viscous stress tensor, $\mathbf{l}'$ is the vector of the inward normal to the free boundary, $\mathbf{n}'$ is its unit vector and K is its mean curvature, which can be calculated from the formula

$$K = \frac{-(1+\xi_{x_1}'^2)\xi_{x_2x_2}' - (1+\xi_{x_2}'^2)\xi_{x_1x_1}' + 2\xi_{x_1}'\xi_{x_2}'\xi_{x_1x_2}'}{2(1+\xi_{x_1}'^2+\xi_{x_2}'^2)^{3/2}}$$

On the solid wall $x_3 = 1$ the boundary conditions have the form

$$\mathbf{v}' = 0, \quad \frac{\partial T'}{\partial x_3} + B_0 T' = \delta_2 \tag{1.5}$$

Problem (1.1)–(1.5) contains the following dimensionless parameters

$$\varepsilon = \beta A h, \quad \omega = \frac{\tilde{\omega} h^2}{\nu}, \quad \mathrm{Re} = \frac{ah}{\nu}, \quad \mathrm{Ga} = \frac{gh^3}{\nu^2}, \quad \mathrm{Pr} = \frac{\nu}{\chi}$$

$$\mathrm{Ma} = \frac{A\sigma_T h^2}{\rho_0 \chi \nu}, \quad C = \frac{\sigma_0 h}{\rho_0 \nu^2}, \quad \mathrm{Bi} = \frac{b_1 h}{k_1}, \quad B_0 = \frac{b_2 h}{k_2}$$

Here ε is the Boussinesq parameter, ω is the dimensionless vibration frequency, Re is the vibration Reynolds number, Ga is the Galileo number, Pr is the Prandtl number, Ma is the Marangoni number, C is the dimensionless surface tension coefficient and Bi, Bo are the Biot numbers.

2. THE ASYMPTOTIC FORM OF HIGH FREQUENCIES

Averaging. We will henceforth be concerned with the asymptotic form of the solution of problem (1.1)–(1.5) in the case when the frequency ω is high, while the vibration Reynolds number is finite: $\mathrm{Re} = O(1)$, $\omega \to \infty$. We can then assume that the following conditions are satisfied for the dimensional frequency $\tilde{\omega}$

$$\frac{h}{c} \leqslant \frac{2\pi}{\tilde{\omega}} \leqslant \min\left(\frac{h^2}{\nu}, \frac{h^2}{\chi}\right)$$

where c is the velocity of sound. The upper limitation denotes that the vibration period must be much less than the characteristic times of action of the viscosity and the thermal conductivity. Breakdown of the left-hand inequality denotes that we must taken into account the compressibility of the liquid, while breakdown of the right-hand inequality means that we must take into account the vibration boundary layers. Estimates show that a range of the frequencies $\tilde{\omega}$ exists for which the above-mentioned conditions are compatible.

We will apply the Krylov–Bogolyubov averaging method to problem (1.1)–(1.5) in the same way as before [1]. In the addition to the slow time t we will introduce the fast time $\tau = \omega t$. We will seek an asymptotic solution as $\omega \to \infty$ in the form of the sum of smooth and fast components, having zero mean with respect to the time τ

$$\mathbf{v}' = \mathbf{v}(x,t) + \tilde{\mathbf{v}}(x,t,\tau), \quad p' = p(x,t) + \omega\tilde{p}(x,t,\tau)$$

$$T' = T(x,t) + \frac{1}{\omega}\tilde{T}(x,t,\tau), \quad \xi' = \xi(x_1,x_2,t) + \frac{1}{\omega}\tilde{\xi}(x_1,x_2,t,\tau) \tag{2.1}$$

Equations for the fast unknowns are obtained after substituting expressions (2.1) into Eqs (1.1) and separating the principal vibration terms as $\omega \to \infty$. As a result we obtain the following system

$$(1-\varepsilon T)\frac{\partial \tilde{\mathbf{v}}}{\partial \tau} = -\nabla\tilde{p} + \mathrm{Re}(1-\varepsilon T)\cos\tau \cdot \mathbf{s}, \quad \operatorname{div}\tilde{\mathbf{v}} = 0 \tag{2.2}$$

$$\frac{\partial \tilde{T}}{\partial \tau} + (\tilde{\mathbf{v}}, \nabla T) = 0 \tag{2.3}$$

We will now consider the boundary conditions. The principal vibration terms in the kinematic boundary condition (1.2) give an equation for the fast component of the free boundary

$$x_3 = \xi(x_1, x_2, t): \ \frac{\partial \tilde{\xi}}{\partial \tau} = -\tilde{v}_1 \frac{\partial \xi}{\partial x_1} - \tilde{v}_2 \frac{\partial \xi}{\partial x_2} + \tilde{v}_3 \tag{2.4}$$

We will assume that, in the dynamic boundary condition (1.3), the dimensionless parameters are independent of ω. We then obtain from (1.3) the following boundary condition for the pulsation pressure

$$x_3 = \xi(x_1, x_2, t): \ \tilde{p} = 0 \tag{2.5}$$

Since the order of Eq. (2.2) is lower than the order of the corresponding equation in system (1.1), for the pulsation velocity $\tilde{\mathbf{v}}$ on the solid wall $x_3 = 1$ we formulate a boundary condition similar to that of an ideal fluid

$$x_3 = 1: \ v_n = 0 \tag{2.6}$$

It is easy to show that the solution of problem (2.2)–(2.6), which is 2π-periodic with respect to τ and which has zero mean, can be written in the form

$$\tilde{\mathbf{v}} = \operatorname{Re} \mathbf{w}(x,t) \sin \tau, \quad \tilde{p} = \operatorname{Re} \Phi(x,t) \cos \tau, \quad \tilde{T} = \operatorname{Re}(\mathbf{w}, \nabla T) \cos \tau$$
$$\tilde{\xi} = -\operatorname{Re}(\mathbf{w}, \mathbf{l}) \cos \tau, \quad \mathbf{l} = \left(-\frac{\partial \xi}{\partial x_1}, -\frac{\partial \xi}{\partial x_2}, 1 \right) \tag{2.7}$$

Here $\mathbf{w}(x, t)$ and $\Phi(x, t)$ are the amplitudes of the pulsation velocity and pressure for which we have the problem

$$(1 - \varepsilon T)\mathbf{w} = -\nabla \Phi + (1 - \varepsilon T)\mathbf{s}, \quad \operatorname{div} \mathbf{w} = 0 \tag{2.8}$$
$$x_3 = \xi(x_1, x_2, t): \ \Phi = 0; \ \ x_3 = 1: \ w_n = 0$$

Formulae (2.7) give expressions for the fast components in terms of the smooth components. Substituting expressions (2.7) into (2.1) and then substituting the expressions obtained into system (1.1)–(1.5), averaging over the fast time τ and retaining terms of the order of unity as $\omega \to \infty$, we obtain a closed autonomous system for the mean values: the unknowns $\mathbf{v}$, q and T. As a result of averaging a vibration-induced mass force [10]

$$\mathbf{F}_v = \frac{1}{2} \operatorname{Re}^2 (\mathbf{w}, \nabla) \nabla \Phi$$

appeared in the equation of motion and vibration-induced stresses, proportional to the square of the vibration Reynolds number, appeared in the dynamic boundary condition.

In the averaged system we take the limit as $\varepsilon \to 0$, assuming that $T = O(1)$ and $\xi = O(1)$, and retain the principal terms. We arrive at the problem

$$\frac{d\mathbf{v}}{dt} = -\nabla q + \Delta \mathbf{v} - \operatorname{Gr} T \gamma + \frac{\operatorname{Re}^2}{2} \varepsilon (\mathbf{w} \wedge \operatorname{rot} T(\mathbf{w} - \mathbf{s}) + (\mathbf{w}, \nabla) T(\mathbf{w} - \mathbf{s}))$$
$$\frac{dT}{dt} = \operatorname{Pr}^{-1} \Delta T, \quad \operatorname{div} \mathbf{v} = 0 \tag{2.9}$$
$$(1 - \varepsilon T)\mathbf{w} = -\nabla \Phi + (1 - \varepsilon T)\mathbf{s}, \quad \operatorname{div} \mathbf{w} = 0$$
$$q = p - \operatorname{Ga} z - \frac{\operatorname{Re}^2}{4} \mathbf{w}^2, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}, \nabla)$$

The boundary conditions have the form

$$x_3 = \xi(x_1, x_2, t): \ (\mathbf{v}, \mathbf{l}) = \frac{\partial \xi}{\partial t}, \quad \mathbf{l} = \left(-\frac{\partial \xi}{\partial x_1}, -\frac{\partial \xi}{\partial x_2}, 1 \right)$$

$$\tau_{ik}n_k - \left(q + \text{Ga}\,\xi - \frac{\text{Re}^2}{2}\left(\frac{\mathbf{w}^2}{2} + \frac{\partial\Phi}{\partial z}(\mathbf{w},\mathbf{l})\right)\right)n_i =$$
$$= 2K\left(C - \frac{\text{Ma}}{\text{Pr}}T\right)n_i + \frac{\text{Ma}}{\text{Pr}}\frac{\partial T}{\partial x_i}, \quad i = 1,2 \tag{2.10}$$
$$\tau_{3k}n_k - \left(q + \text{Ga}\,\xi - \frac{\text{Re}^2}{2}\left(\frac{\mathbf{w}^2}{2} + \frac{\partial\Phi}{\partial z}(\mathbf{w},\mathbf{l})\right)\right)n_3 = 2K\left(C - \frac{\text{Ma}}{\text{Pr}}T\right)n_3$$
$$\frac{\partial T}{\partial n} - \text{Bi}\,T = \delta_1, \quad \Phi = 0$$
$$x_3 = 1: \quad \mathbf{v} = 0, \quad \frac{\partial T}{\partial x_3} + B_0 T = \delta_2, \quad w_n = 0$$

3. THE EQUILIBRIUM SOLUTION AND ITS STABILITY

We will investigate problem (2.9), (2.10) further. We will assume that the heat-exchange conditions are specified in such a way that an equilibrium solution exists with a linear temperature profile

$$\mathbf{v}_0 = 0, \quad T_0 = z, \quad q_0 = -\text{Gr}\frac{z^2}{2} + \frac{\text{Re}^2}{4}\cos^2\varphi$$
$$\mathbf{w}_0 = (\cos\varphi, 0, 0), \quad \Phi_0 = \left(z - \varepsilon\frac{z^2}{2}\right)\sin\varphi, \quad \xi_0 = 0 \tag{3.1}$$

We will investigate its stability by the linearization method, assuming

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \quad q = q_0 + q_1, \quad T = T_0 + T_1$$
$$\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1, \quad \Phi = \Phi_0 + \Phi_1, \quad \xi = \xi_0 + \xi_1$$

Assuming the perturbations to be infinitesimal and proportional to exp λt, we obtain the following eigenvalue problem for the corresponding amplitudes

$$\lambda\mathbf{v} = -\nabla q + \Delta\mathbf{v} - \text{Gr}\,T\boldsymbol{\gamma} - \frac{\text{Re}^2}{2}\varepsilon\sin\varphi\, w_3\boldsymbol{\gamma}$$
$$\lambda T = \text{Pr}^{-1}\Delta T - v_3, \quad \text{div}\,\mathbf{v} = 0 \tag{3.2}$$
$$(1 - \varepsilon z)\mathbf{w} = -\nabla\Phi - \varepsilon\sin\varphi\, T\boldsymbol{\gamma}, \quad \text{div}\,\mathbf{w} = 0$$

After linearization, the boundary conditions take the form

$$x_3 = 0: \quad v_3 = \lambda\xi, \quad \tau_{i3} = \frac{\text{Ma}}{\text{Pr}}\left(\frac{\partial T}{\partial x_i} + \frac{\partial\xi}{\partial x_i}\right), \quad i = 1,2$$
$$\tau_{33} - q = -C\Delta_1\xi + \text{Ga}\,\xi - \frac{\text{Re}^2}{2}\sin\varphi\, w_3 \tag{3.3}$$
$$\Phi + \sin\varphi\,\xi = 0, \quad \frac{\partial T}{\partial x_3} - \text{Bi}(T + \xi) = 0 \quad \left(\Delta_1 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$
$$x_3 = 1: \quad \mathbf{v} = 0, \quad \frac{\partial T}{\partial x_3} + B_0 T = 0, \quad w_3 = 0$$

We can conclude from the form of eigenvalue problem (3.2), (3.3) that horizontal ($\varphi = 0$) high-frequency vibrations have no effect on the onset of convection in a thin liquid layer with a deformable free boundary.

We will further consider the case when $\varphi \neq 0$, which enables us to make the following replacement

$$\Phi = \sin\varphi\Psi, \quad w_3 = \sin\varphi W_3$$

Eliminating the pressure q and the horizontal components υ_1, υ_2 and w_1, w_2, we arrive at the problem

$$\begin{aligned}
&\lambda\Delta\upsilon_3 = \Delta^2\upsilon_3 - \mathrm{Gr}\,\Delta_1 T - \mu_s\varepsilon\Delta_1 W_3, \quad \lambda T = \mathrm{Pr}^{-1}\Delta T - \upsilon_3\\
&\Delta\Psi = \varepsilon\left(W_3 - \frac{\partial T}{\partial x_3}\right), \quad (1-\varepsilon z)W_3 = -\frac{\partial\Psi}{\partial x_3} - \varepsilon T\\
&x_3 = 0:\ \upsilon_3 = \lambda\xi, \ -\frac{\partial^2\upsilon_3}{\partial x_3^2} + \Delta_1\upsilon_3 = \frac{\mathrm{Ma}}{\mathrm{Pr}}\Delta_1(T+\xi)\\
&2\Delta_1\frac{\partial\upsilon_3}{\partial x_3} - \lambda\frac{\partial\upsilon_3}{\partial x_3} + \Delta\frac{\partial\upsilon_3}{\partial x_3} = -C\Delta_1^2\xi + \mathrm{Ga}\,\Delta_1\xi - \mu_s\Delta_1 W_3\\
&\frac{\partial T}{\partial x_3} - \mathrm{Bi}(T+\xi) = 0, \quad \Psi + \xi = 0\\
&x_3 = 1:\ \upsilon_3 = \frac{\partial\upsilon_3}{\partial x_3} = 0, \ \frac{\partial T}{\partial x_3} + B_0 T = 0, \ W_3 = 0
\end{aligned} \tag{3.4}$$

Here

$$\mu_s = \frac{(\mathrm{Re}\sin\varphi)^2}{2} = \frac{1}{2}\left(\frac{ah\sin\varphi}{\nu}\right)^2$$

Hence, when $\varphi \neq 0$ the action of a high-frequency vibration is characterized by the single parameter μ_s, which includes both the direction φ and the amplitude a of the vibration velocity, unlike Rayleigh–Benard vibration convection, which is characterized by two vibration parameters – the velocity and the direction of the vibration [3, 17]. Moreover, in the case of Rayleigh–Benard convection, horizontal vibrations amplify thermogravitational convection [3, 17].

We will introduce normal perturbations, assuming

$$(\upsilon_3, T, \Psi, W_3, \xi) = \exp(i\alpha_1 x_1 + i\alpha_2 x_2)(\mathrm{Pr}^{-1}\upsilon(z), \theta(z), \Phi(z), w(z), \delta)$$

Then, eigenvalue problem (3.4) takes the form

$$\begin{aligned}
&\lambda L\upsilon = L^2\upsilon + \mathrm{Gr}\,\mathrm{Pr}\,\alpha^2\theta + \mu_s\varepsilon\,\mathrm{Pr}\,\alpha^2 w, \quad \lambda\,\mathrm{Pr}\,\theta = L\theta - \upsilon\\
&z = 0:\ \upsilon = \lambda\,\mathrm{Pr}\,\delta, \ D^2\upsilon + \alpha^2\upsilon = \mathrm{Ma}\,\alpha^2(\theta+\delta)\\
&(3\alpha^2+\lambda)D\upsilon - D^3\upsilon = \mathrm{Pr}\,\alpha^2((C\alpha^2+\mathrm{Ga})\delta - \mu_s w)\\
&D\theta - \mathrm{Bi}(\theta+\delta) = 0\\
&z = 1:\ \upsilon = D\upsilon = 0, \ D\theta + B_0\theta = 0
\end{aligned} \tag{3.5}$$

The functions w and Φ are the solutions of the problem

$$\begin{aligned}
&L\Phi = \varepsilon(w - D\theta), \quad (1-\varepsilon z)w = -D\Phi - \varepsilon\theta\\
&z = 0:\quad \Phi + \delta = 0; \quad z = 1:\quad w = 0
\end{aligned} \tag{3.6}$$

We will separate the principal terms as $\varepsilon \to 0$ in the expression for $\Phi(z)$ and $w(z)$, assuming

$$\Phi = \Phi_0 + \varepsilon\Phi_1 + \ldots, \quad w = w_0 + \varepsilon w_1 + \ldots$$

Substituting these expansions into (3.6), we obtain the following problems

$$L\Phi_0 = 0, \quad w_0 = -D\Phi_0; \quad \Phi_0(0) = -\delta, \quad w_0(1) = 0 \tag{3.7}$$

$$L\Phi_1 = w_0 - D\theta, \quad w_1 = -D\Phi_1 - \theta + z w_0; \quad \Phi_1(0) = 0, \quad w_1(1) = 0 \tag{3.8}$$

Problem (3.7) has the solution

$$\Phi_0(z) = \delta(\operatorname{th}\alpha \operatorname{sh}\alpha z - \operatorname{ch}\alpha z), \quad w_0(z) = -\delta\alpha(\operatorname{th}\alpha \operatorname{ch}\alpha z - \operatorname{sh}\alpha z)$$

Retaining principal terms in ε, as $\varepsilon \to 0$ we can write problem (3.5) in the form

$$\begin{aligned}
&\lambda L v = L^2 v + \mathrm{Ra}\,\alpha^2\theta + R_\upsilon \alpha^2 (w_0 + \varepsilon w_1), \quad \lambda \Pr\theta = L\theta - v \\
&w_1 = -D\Phi_1 - \theta + z w_0, \quad L\Phi_1 = w_0 - D\theta \\
&z = 0: \quad v = \lambda \Pr\delta, \quad D^2 v + \alpha^2 v = \mathrm{Ma}\,\alpha^2(\theta + \delta) \\
&(3\alpha^2 + \lambda) D v - D^3 v = \Pr\alpha^2 (C\alpha^2 + \mathrm{Ga} + \mu_s \alpha \operatorname{th}\alpha)\delta - R_\upsilon w_1 \\
&D\theta - \mathrm{Bi}(\theta + \delta) = 0, \quad \Phi_1 = 0 \\
&z = 1: \quad v = Dv = 0, \quad D\theta + B_0\theta = 0, \quad w_1 = 0
\end{aligned} \tag{3.9}$$

Here $\mathrm{Ra} = \mathrm{Ga}\varepsilon\Pr$ is the gravitational Rayleigh number and $R_\upsilon = \mu_s\varepsilon\Pr$ is the vibration Rayleigh number.
We can further consider different types of convective instability in a layer with a deformable free boundary.

Thermocapillary convection in a layer of uniform liquid. We will consider Marangoni convection in a thin layer of uniform liquid, assuming $\varepsilon = 0$, $\mathrm{Ra} = 0$, $R_\upsilon = 0$. The following eigenvalue problem corresponds to this case

$$\begin{aligned}
&\lambda L v = L^2 v, \quad \lambda \Pr\theta = L\theta - v \\
&z = 0: \quad v = \lambda \Pr\delta, \quad D^2 v + \alpha^2 v = \mathrm{Ma}\,\alpha^2(\theta + \delta) \\
&(3\alpha^2 + \lambda) D v - D^3 v = \Pr\alpha^2 (C\alpha^2 + \mathrm{Ga} + \mu_s \alpha \operatorname{th}\alpha)\delta \\
&D\theta - \mathrm{Bi}(\theta + \delta) = 0 \\
&z = 1: \quad v = Dv = 0, \quad D\theta + B_0\theta = 0
\end{aligned} \tag{3.10}$$

This problem has been investigated in the case when $\mu_s = 0$ in numerous papers (see the review [18]). It has been shown that the deformation of the free boundary has a considerable influence on the long-wave ($\alpha \to 0$) instability. Moreover it has been established that as the surface tension C increases the free boundary is smoothed out and the critical Marangoni numbers approach values corresponding to a non-deformable free boundary. In the case considered here we can achieve this effect by increasing the vibration parameter μ_s, since it enhances the effect of surface tension

$$C_s = C + \mathrm{Ga}/\alpha^2 + \mu_s \operatorname{th}\alpha/\alpha$$

The contribution of vibration may be comparable with the contribution of gravitational forces. For a water layer of thickness $h = 1$ mm, a vibration frequency $\omega = 100$ Hz and an amplitude $a/\omega = 1$ mm, we have $T = 7 \times 10^4$, $\mathrm{Ga} = 10^4$ and $\mu_s = 0.5 \times 10^4 \sin^2\varphi$.
In order to confirm these conclusions we will present some asymptotic and numerical results.

The long-wave asymptotic form. We will consider monotonic instability. Assuming $\lambda = 0$ in system (3.10) and eliminating the function $\upsilon(z)$, we obtain the problem

$$\begin{aligned}
&L^3\theta = 0 \\
&z = 0: \quad L\theta = 0, \quad L^2\theta = \mathrm{Ma}\,\alpha^2(\theta + \delta) \\
&\mathrm{Cr}(3\alpha^2 DL\theta - D^3 L\theta) = \alpha^2(\alpha^2 + \mathrm{BO} + \mu\alpha \operatorname{th}\alpha)\delta \\
&D\theta - \mathrm{Bi}(\theta + \delta) = 0 \\
&z = 1: \quad L\theta = DL\theta = 0, \quad D\theta + B_0\theta = 0
\end{aligned} \tag{3.11}$$

Here $\mathrm{Cr} = (\Pr C)^{-1}$ is the capillary parameter, $\mathrm{BO} = \mathrm{Ga}/C$ is the gravitational Bond number and $\mu = \mu_s/C$ is the vibration Bond number.
We will investigate the behaviour of the eigenvalues $\mathrm{Ma}(\alpha)$ as $\alpha \to 0$ by the perturbation method. Expanding the unknowns in series in powers of α^2

$$\theta = \theta_0 + \alpha^2\theta_1 + \ldots, \quad \delta = \delta_0 + \alpha^2\delta_1 + \ldots, \quad \mathrm{Ma} = \mathrm{Ma}_0\,\alpha^{-2} + \mathrm{Ma}_1 + \mathrm{Ma}_2\,\alpha^2 + \ldots$$

we obtain boundary-value problems for the coefficients. By solving these problems we obtain the principal terms of the asymptotic form of the Marangoni number

$$\mathrm{Ma}(\alpha) = \frac{48}{1 + 72\,\mathrm{Cr/BO}}\left(\frac{\mathrm{Bi}}{\alpha^2} + 1\right) + O(\alpha^2), \quad B_0 = 0 \tag{3.12}$$

$$\mathrm{Ma}(\alpha) = \frac{2}{3}\frac{\mathrm{BO}}{\mathrm{Cr}}(1+\mathrm{Bi}) + \frac{2}{3\mathrm{Cr}}\left((1+\mathrm{Bi})\left(1+\mu-\frac{2\,\mathrm{BO}}{15}\right)+\frac{\mathrm{BO}}{3}-\frac{(1+\mathrm{Bi})\mathrm{BO}^2}{120Cr}\right)\alpha^2 + O(\alpha^4),$$

$$B_0 = \infty \tag{3.13}$$

Numerical results. Problem (3.10) was solved by reduction to a transcendental equation, which was constructed analytically and numerically by the ranging method. We chose the Marangoni number and the frequency of neutral oscillations c as the required parameters and assumed $\lambda = ic$. The results obtained were compared with the asymptotic and known values when $\mu = 0$. We investigated monotonic and oscillatory instability.

The existence of oscillatory instability was previously found for $\mathrm{Ma}(\alpha) < 0$ as $\alpha \to 0$ [11] when Ma $(\alpha) > 0$, $\alpha \to \infty$ [18]. However, as calculations show, when Ma > 0 the first loss of stability is monotonic. The table shows values of $\mathrm{Ma}(\alpha)$ for $c = 0$, $B_0 = 0$, Bi = 0.1, Cr = 0.033 and BO = 0.0049 (glycerin, h = 0.1 mm) as a function of the parameter μ. The asymptotic form (3.12) is satisfied when $\alpha \to 0$.

In Fig. 1 we show neutral curves of $\mathrm{Ma}(\alpha)$ of monotonic instability (the continuous curves) and oscillatory instability (the dashed curves). In Fig. 2 we show neutral curves of the oscillatory instability

Table 1

α	$\mu = 0$	$\mu = 3.3$	$\mu = 33$	$\mu = 3.3 \times 10^2$	$\mu = 3.3 \times 10^4$	Cr = 0
5.00	206.71	207.21	207.80	207.95	207.97	207.97
4.00	137.72	139.23	140.73	141.06	141.10	141.10
3.00	86.66	90.68	93.96	94.61	94.69	94.70
2.50	66.84	73.17	77.91	78.81	78.92	78.92
2.00	49.59	59.31	66.22	67.50	67.65	67.65
1.50	33.76	47.84	58.34	60.32	60.56	60.56
1.00	18.92	36.53	53.91	57.74	58.21	58.21
0.50	6.98	21.46	54.15	67.90	69.92	69.94
0.10	3.31	10.46	67.07	308.44	526.16	529.95
0.05	6.12	12.89	71.79	510.90	1.915×10^3	1.970×10^3
0.04	8.32	15.05	74.25	559.96	2.919×10^3	3.050×10^3
0.03	13.11	19.80	79.24	608.41	4.985×10^3	5.383×10^3
0.02	26.82	33.48	93.09	658.23	1.021×10^4	1.205×10^4
0.01	100.90	107.55	167.27	756.37	2.795×10^4	4.805×10^4
10^{-3}	9.881×10^3	9.887×10^3	9.947×10^3	1.054×10^4	7.537×10^4	4.800×10^6
10^{-4}	9.879×10^5	9.879×10^5	9.879×10^5	9.885×10^5	1.054×10^6	4.800×10^8
10^{-5}	9.879×10^7	9.879×10^7	9.879×10^7	9.879×10^7	9.885×10^7	4.800×10^{10}

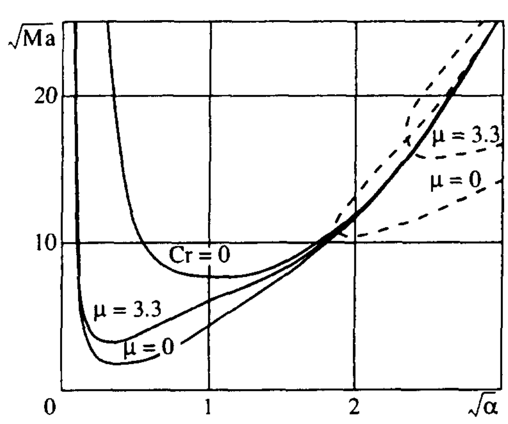

Fig. 1

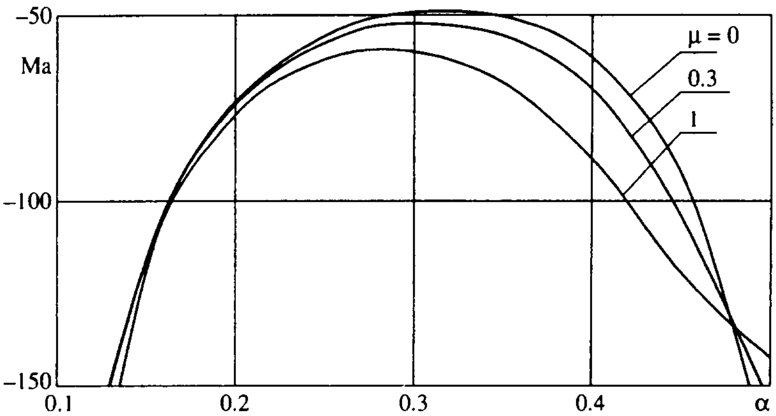

Fig. 2

when Bi = 0, Cr = 0.01, Pr = 0.01, BO = 0 and $B_0 = \infty$. When $\mu = 0$ the results of the calculations agree with those obtained earlier in [19].

Thus, the numerical results show that when $\varphi \neq 0$, high-frequency vibration smooths out the free boundary, and the Marangoni numbers approach the values obtained previously in [20].

Pearson vibration convection. We will now assume that the free boundary is non-deformable on average ($\delta = 0$) and that the liquid is non-uniform ($\varepsilon \neq 0$). In this case problem (3.9) has the form

$$\begin{aligned}
&\lambda L v = L^2 v + \mathrm{Ra}\,\alpha^2\theta - \bar{\mu}\alpha^2(D\Phi_1 + \theta)\\
&\lambda \Pr\theta = L\theta - v, \quad L\Phi_1 = -D\theta\\
&z = 0: \quad v = 0, \quad D^2 v - \mathrm{Ma}\,\alpha^2\theta = 0, \quad D\theta - \mathrm{Bi}\,\theta = 0, \quad \Phi_1 = 0 \qquad (3.14)\\
&z = 1: \quad v = Dv = 0, \quad D\theta + B_0\theta = 0, \quad D\Phi_1 + \theta = 0
\end{aligned}$$

Here $\mu = \mu_s\varepsilon^2\Pr = R_\upsilon\varepsilon$ is the vibration parameter.

Assuming Ra = 0, we obtain the relation between the critical Marangoni number and this parameter

$$\mathrm{Ma}_*(\bar{\mu}) = \min_\alpha \mathrm{Ma}(\alpha, \bar{\mu})$$

Calculations showed that the instability is monotonic and the values of Ma_* increase as the parameter $\bar{\mu}$ increases.

We will represent the parameter $\bar{\mu}$ in the form

$$\bar{\mu} = \mathrm{Ma}^2 s^2, \quad s^2 = (a^2\beta^2\rho_0\chi\nu\sin^2\varphi)/(2\sigma_0^2)$$

where s is the dimensionless vibration velocity. Calculations show that values $s_*(B_0, \mathrm{Bi})$ exist such that when $s > s_*$ there is absolute stability. For example, $s_*(\infty, 0.1) = 0.26$, $s_*(0, 0.1) = 0.95$. Similar conclusions can be reached for thermogravitational convection, assuming Ma = 0 in problem (3.14) and considering the relation $\mathrm{Ra}_*(r)$, where $r^2 = \bar{\mu}/\mathrm{Ra}^2$. We obtain that $r_*(\infty, 0.1) = 0.029$.

The interaction of the thermogravitational and thermocapillary mechanisms of instability, when there is vibration, can be tracked by investigating the dependence of the critical Rayleigh numbers

$$\mathrm{Ra}_*(\mathrm{Ma}, \bar{\mu}) = \min_\alpha \mathrm{Ra}(\alpha, \mathrm{Ma}, \bar{\mu})$$

Calculations showed that the instability is monotonic while the neutral curves $\mathrm{Ra}_*(\mathrm{Ma})$ for large values of $\bar{\mu}$ are close to straight lines

$$\mathrm{Ra}_* = k\,\mathrm{Ma} + b(\bar{\mu})$$

where $k = -8.97$, $b(10^3) = 1232$, $b(5 \times 10^3) = 2536$, $b(10^4) = 3551$. As the parameter $\bar{\mu}$ increases the boundary of stability departs to infinity.

If we take the Oberbek–Boussinesq equations in the initial model, then instead of problem (3.14) we obtain the problem

$$\lambda L v = L^2 v + \mathrm{Ra}\,\alpha^2\theta - \bar{\mu}(\alpha^2 \sin\varphi(D\Phi + \theta\sin\varphi) + i\alpha\cos\varphi\Phi)$$

$$\lambda\,\mathrm{Pr}\,\theta = L\theta - v, \quad L\Phi = -i\alpha\cos\varphi\theta - \sin\varphi D\theta$$

$$z = 0: \quad v = 0, \quad D^2 v - \mathrm{Ma}\,\alpha^2\theta = 0, \quad D\theta - \mathrm{Bi}\,\theta = 0, \quad \Phi = 0$$

$$z = 1: \quad v = Dv = 0, \quad D\theta + B_0\theta = 0, \quad D\Phi + \sin\varphi\theta = 0$$

When $\varphi = \pi/2$ these problems are identical. If $\varphi \neq \pi/2$, the solutions $\mathrm{Ma}(\alpha)$ are qualitatively different. The fact that the problems are identical when $\varphi = \pi/2$ is natural – in this case the transfer acceleration $\mathbf{w}_e$ in (1.1) can be included in the pressure from the beginning and one can use the Oberbek–Boussinesq equations.

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